Public Economics

Problem set 3

Solutions

Marc Sangnier - marc.sangnier@univ-amu.fr

Exercise 1

Consider a representative consumer that has the following utility function:

$$U(x_1, x_2, l) = \frac{1}{\alpha} x_1^{\alpha} + \frac{1}{\beta} x_2^{\beta} - l,$$

where $(\alpha, \beta) \in (0,1)^2$, x_i denotes consumption of good i and l is labor time. Hourly wage is $w = 1$, such that the consumer's budget constraint is simply:

$$x_1 q_1 + x_2 q_2 \leq l,$$

where q_i is the unit price of good i payed by the consumer.

1. Determine the consumer's demand in both goods. Calculate ε_1 and ε_2, the price-elasticities of goods.

Using the budget constraint, we can rewrite the utility function as:

$$U(x_1, x_2) = \frac{1}{\alpha} x_1^{\alpha} + \frac{1}{\beta} x_2^{\beta} - x_1 q_1 - x_2 q_2.$$

By taking the first order condition condition with respect to x_1, we get:

$$x_1^{\alpha-1} - q_1 = 0 \Leftrightarrow x_1 = q_1^{\frac{1}{\alpha-1}}.$$

Similarly, we obtain:

$$x_2 = q_2^{\frac{1}{\beta-1}}.$$

The price-elasticity of good 1 is:

$$\varepsilon_1 = \frac{\partial x_1}{\partial q_1} \frac{q_1}{x_1} = \frac{1}{\alpha - 1} q_1^{\frac{1}{\alpha-1}-1} \frac{q_1}{q_1^{\frac{1}{\alpha-1}}} = \frac{1}{\alpha - 1}.$$

Similarly, we get:

$$\varepsilon_2 = \frac{1}{\beta - 1}.$$

For some (good) reason, the government want to set taxes on consumption goods. She chooses to set a unit tax t_i on good i, such that $q_i = p_i + t_i$. From now on, assume that $\alpha = \frac{1}{2}$ and $\beta = \frac{2}{3}$. Also assume that producers' unit revenues are such that $p_1 = p_2 = 1$.

2. Use the inverse elasticity rule to show that optimal taxes will be such that:

$$t_2 = 2\frac{1}{1+\frac{3}{t_1}}.$$

> According to the inverse elasticity rule, we know that optimal taxes will be such that:
> $$\frac{p_1+t_1}{t_1}\frac{1}{\varepsilon_1} = \frac{p_2+t_2}{t_2}\frac{1}{\varepsilon_2}.$$
> Using the expressions of price-elasticities from the previous question and since $\alpha = \frac{1}{2}$, $\beta = \frac{2}{3}$, we know that $\varepsilon_1 = -2$ and $\varepsilon_2 = -3$. Thus, the previous expression can be rewritten as:
> $$\frac{1+t_1}{t_1}\frac{1}{2} = \frac{p_2+t_2}{t_2}\frac{1}{3} \Leftrightarrow t_2 = 2\frac{1}{1+\frac{3}{t_1}}.$$

3. Which unit-tax is the largest? Why was it to be expected?

> A quick inspection of the previous expression reveals that $t_2 < t_1$. This was to be expected as one of the message of the inverse elasticity rule is that we should tax more goods that have less elastic demands. Here, we know that the demand for good 2 is more elastic that the one for good 1. Thus, the latter will logically be more taxed.

4. Determine t_1 and t_2 such that the government has a total revenue of $R = 0.375$. *Note:* $x = 0.6$ is the solution the following equation:

$$0.375 \approx x(1+x)^{-2} + \left(2\frac{1}{1+\frac{3}{x}}\right)\left(1+2\frac{1}{1+\frac{3}{x}}\right)^{-3}.$$

> The government must set taxes such as her total revenue is R, i.e. t_1 and t_2 must be set such that:
> $$R \leq t_1x_1 + t_2x_2.$$
> However, x_i depends on t_i. So, the government must in fact set taxes such that:
> $$R \leq t_1(1+t_1)^{\frac{1}{\alpha-1}} + t_2(1+t_2)^{\frac{1}{\beta-1}}.$$
> Since we know from the inverse elasticity rule that there is a relation between t_1 and t_2, the expression we need to solve for t_1 is (simply):
> $$R = t_1(1+t_1)^{-2} + \left(2\frac{1}{1+\frac{3}{t_1}}\right)\left(1+2\frac{1}{1+\frac{3}{t_1}}\right)^{-3}.$$

For $R \approx 0.375$, the solution of this equation is $t_1 = 0.6$. As a consequence, we must set:

$$t_2 = 2\frac{1}{1+\frac{3}{0.6}} \approx 0.33.$$

As expected, the good that has the more elastic demand is taxed less than the other one.

Exercise 2

Consider an individual that is isolated within the society. Her utility function is:

$$U(c,l) = c - \frac{1}{1+\mu} l^{1+\mu},$$

where $\mu > 0$, c denotes consumption, and l is labor time. Hourly wage is $w = 1$, such that the budget constraint is simply:

$$c \leq l(1-t),$$

where t is the tax rate faced by the individual.

1. Determine l^*, the individual's optimal labor supply given a tax rate t.

The utility function can be rewritten as:

$$U(l) = l(1-t) - \frac{1}{1+\mu} l^{1+\mu}.$$

The first order condition with respect to l is thus:

$$(1-t) - l^{\mu} = \Leftrightarrow l^*(1-t)^{\frac{1}{\mu}}.$$

2. Determine ε, the individual's labor supply elasticity with respect to net of tax wage.

$$\varepsilon = \frac{\partial l^*}{\partial(1-t)} \frac{1-t}{l^*} = \frac{1}{\mu}.$$

So, the labor supply elasticity is decreasing with μ.

Government's revenue from taxing the individual is $R = l^* t$. Let us assume that the social welfare function used by the government is:

$$\mathbb{W} = \alpha R + U(l^*),$$

where $\alpha > 1$.

3. Interpret α.

α is the marginal benefit for the government to raise more taxes from this individual. It can be interpreted as the relative weight put by the government on beneficiaries of public expenditure relatively to this individual's utility.

4. Show that the government will choose:

$$t^* = \frac{(\alpha-1)\mu}{(\alpha-1)(1+\mu)+1}.$$

Let us look for t that maximizes :

$$\mathbb{W} = \alpha(1-t)^{\frac{1}{\mu}}t + (1-t)(1-t)^{\frac{1}{\mu}} - \frac{1}{1+\mu}(1-t)^{\frac{1+\mu}{\mu}},$$

which can be rewritten as:

$$\mathbb{W} = \alpha(1-t)^{\frac{1}{\mu}}\left\{1+t(\alpha-1)\right\} - \frac{1}{1+\mu}(1-t)^{\frac{1+\mu}{\mu}}.$$

The first order condition is:

$$\frac{\partial\mathbb{W}}{\partial t} = (1-t)^{\frac{1}{\mu}}(\alpha-1) - \left\{1+t(\alpha-1)\right\}\frac{1}{\mu}(1-t)^{\frac{1}{\mu}-1} + \frac{1}{1+\mu}\frac{1+\mu}{\mu}(1-t)^{\frac{1+\mu}{\mu}-1} = 0.$$

Solving for t yields:

$$t^* = \frac{(\alpha-1)\mu}{(\alpha-1)(1+\mu)+1}.$$

5. Comment on how t^* varies with α and μ.

The derivative of t^* with respect to μ is:

$$\frac{\partial t^*}{\partial\mu} = \frac{\alpha(\alpha-1)}{\left[(\alpha-1)(1+\mu)+1\right]^2}.$$

It is positive as long as $\alpha > 1$. Remember that the labor supply elasticity is decreasing with μ. So, the lower the individual's labor supply elasticity, the higher will be the tax rate. Also note that:

$$\lim_{\mu\to+\infty}\frac{1}{t^*} = 1 \Leftrightarrow \lim_{\mu\to+\infty} t^* = \lim_{\varepsilon\to 0} t^* = 1,$$

which means that the tax rate will be 100% if labor supply elasticity is zero.

The derivative of t^* with respect to α is:

$$\frac{\partial t^*}{\partial\alpha} = \frac{\mu}{\left[(\alpha-1)(1+\mu)+1\right]^2}.$$

It is positive. So, the lower the relative weight put by the government on the individual's utility, the higher will be the tax rate.