# Public Economics

## Problem set 2

### Solutions

Marc Sangnier - marc.sangnier@univ-amu.fr

Exercises are inspired from *Intermediate Public Economics*, by J. Hindriks and G.D. Myles.

## Exercise 1

Let us consider an economy populated by 2 consumers—$A$ and $B$—who are endowed with 1 unit of income and derive utility from the consumption of a private good $x$ and a pure public good $G$. Individual $i$ utility function is given by:

$$U^i = \log(x_i) + \log(G),$$

where $x_i = 1 - g_i$ denotes consumption of the private good by consumer $i$, and $G = g_A + g_B$ is the total quantity public good that is produced from individuals contributions.

1. Determine individual $A$'s private provision of the public good when considering $g_B$ as given.

   Individual $A$'s utility function can be rewritten as:

   $$U^A = \log(1 - g_A) + \log(g_A + g_B).$$

   Maximizing this expression with respect to $g_A$, we get:

   $$g_A = \frac{1}{2} - \frac{g_B}{2}.$$

2. Determine individual $B$'s private provision of the public good when considering $g_A$ as given.

   By using the same reasoning, we get:

   $$g_B = \frac{1}{2} - \frac{g_A}{2}.$$

3. Use the two reaction functions to find $G^*$, the quantity of public good that is supplied at the Nash equilibrium.

The equilibrium contributions $g_A^*$ and $g_B^*$ are solutions of:

$$\begin{cases} g_A = \frac{1}{2} - \frac{g_B}{2}, \\ g_B = \frac{1}{2} - \frac{g_A}{2}. \end{cases}$$

The yields:

$$g_A^* = \frac{1}{3} \text{ and } g_A^* = \frac{1}{3}. \text{ So: } G^* = \frac{2}{3}.$$

4. Determine $\bar{G}$, the efficient level of public good provision. Contrast it with the decentralized equilibrium.

The efficient level of public good provision can be retrieved via Samuelson's rule :

$$\frac{\partial U^A/\partial g_A}{\partial U^A/\partial x_A} + \frac{\partial U^B/\partial g_B}{\partial U^B/\partial x_B} = 1$$

That is:

$$\frac{1 - g_A}{g_A + g_B} + \frac{1 - g_B}{g_A + g_B} = 1.$$

Since both individuals are identical, $g_A = g_B = g$. We can rewrite the above expression as:

$$2\frac{1-g}{2g} = 1 \Leftrightarrow g = \frac{1}{2}. \text{ So: } \bar{G} = 1.$$

It is clear that $\bar{G} > G^*$.

5. Show that producing $\bar{G}$ is Pareto-superior to producing $G^*$.

Under $G^*$, individual $i$'s utility is:

$$U^i_{G^*} = \log(\frac{2}{3}) + \log(\frac{2}{3}) = \log(\frac{4}{9})$$

Under $\bar{G}$, individual $i$'s utility is:

$$U^i_{\bar{G}} = \log(\frac{1}{2}) + \log(1) = \log(\frac{1}{2})$$

Since $U^i_{G^*} < U^i_{\bar{G}}$, both individuals are better off when producing $\bar{G}$.

6. Show that private contribution required to produce $\bar{G}$ cannot be sustained without the intervention of some third party that would be able to constrain individuals' contributions.

Assume we managed to reach the level of production $\bar{G}$, with $g_A = g_B = g = \frac{1}{2}$. Given that individual $B$ is producing $g_B = \frac{1}{2}$, the optimal contribution by $A$ is:

$$g_A = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

At this point, given that individual $A$ is producing $g_A = \frac{1}{4}$, the optimal contribution by $B$ is:

$$g_B = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}.$$

Given that individual $B$ is producing $g_B = \frac{3}{8}$, optimal contribution by $A$ is $\frac{5}{16}$ ...In the absence of any constraint, individuals will continue to adjust until they reach the Nash equilibrium.

## Exercise 2

Let us consider an economy populated by 2 individuals—$A$ and $B$—who consume 2 goods—1 and 2. Individuals' utility function are:

$$U^A = \log(x_1^A) + x_2^A - \frac{1}{2}\log(x_1^B),$$

and,

$$U^B = \log(x_1^B) + x_2^B - \frac{1}{2}\log(x_1^A),$$

where $x_j^i$ is the quantity of good $j$ consumed by individual $i$. Each individual is endowed with 1 unit of income. Let the unit prices of both goods be 1.

1. Calculate the decentralized equilibrium situation of this economy.

Each individual maximizes her utility function subject to her budget constraint. Accordingly, the Lagrangian of individual $i$'s optimization problem is:

$$\mathbb{L} = U^i = \log(x_1^i) + x_2^i - \frac{1}{2}\log(x_1^{-i}) + \lambda_i(1 - x_1^i - x_2^i),$$

where $x_1^{-i}$ denotes consumption of good 1 by the other consumer. Solving this program for each individual yields:

$$x_1^A = 1 \text{ and } x_2^A = 0,$$
$$x_1^B = 1 \text{ and } x_2^B = 0.$$

2. Calculate the social optimum if the social welfare function is the sum of individuals' utility functions.

Let us maximize $\mathbb{W} = U^A + U^B$ with respect to $x_1^A$, $x_1^B$, $x_2^A$, and $x_2^B$, subject to $x_1^A + x_2^B \leq 1$ and $x_1^B + x_2^B \leq 1$. We get :

$$x_1^A = \frac{1}{2} \text{ and } x_2^A = \frac{1}{2},$$
$$x_1^B = \frac{1}{2} \text{ and } x_2^B = \frac{1}{2}.$$

3. Check that the social optimum is Pareto-superior to the decentralized one.

At the decentralized equilibrium, individual $i$'s utility is:

$$U^i = \log(1) + 0 - \frac{1}{2}\log(1) = 0.$$

At the social optimum, individual $i$'s utility is:

$$U^i = \log(\frac{1}{2}) + \frac{1}{2} - \frac{1}{2}\log(\frac{1}{2}) = \frac{1}{2}\left(1 - \log(2)\right).$$

As $\log(2) < 1$, the second expression is larger than the first one. So, both consumers are better off at the social optimum.

4. Show that the social optimum can be reached in a decentralized framework thanks to a tax $t$ placed on good 1 (so, the price of this good is now $1+t$), with the tax revenues returned equally to consumers via a lump-sum transfer $T$.

Individual $i$'s Lagrangian should now be written as:

$$\mathbb{L} = U^i = \log(x_1^i) + x_2^i - \frac{1}{2}\log(x_1^{-i}) + \lambda_i(1 + T - (1+t)x_1^i - x_2^i).$$

Solving yields:

$$x_1^i = \frac{1}{1+t} \text{ and } x_2^i = 1 + T - \frac{1}{1+t}(1+t) = T.$$

Since we want $x_1^i = \frac{1}{2}$, we just need to set $t = 1$. Total tax revenues will thus be $t(x_1^A + x_1^B) = 1$ and $T$ will be equal to $\frac{1}{2}$.