Public Economics

Problem set 1

Solutions

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Exercises are inspired from *Intermediate Public Economics*, by J. Hindriks and G.D. Myles.

Exercise 1

Let us consider the following profiles of preferences over 4 alternatives $\{a, b, c, d\}$ by 5 voters:

	Voter 1	Voter 2	Voter 3	Voter 4	Voter 5
Most preferred option	a	b	d	b	c
	b	a	a	a	a
	c	c	b	c	b
Least preferred option	d	d	c	d	d

In other terms, preferences of voter 2 are such that: $b \succ_2 a \succ_2 c \succ_2 d$.

1. Which will be the chosen alternative under pairwise majority voting? How do we name such an alternative?

   a wins over b (3–2), c (4–1), and d (4–1). This alternative will be chosen under pairwise majority voting. This is the Condorcet winner.

2. Plurality voting is a scoring rule in which each voter's preferred option is given 1 point an all others none. Show that plurality voting does not select the same alternative as pairwise majority voting.

   Under plurality voting, alternative b receives 2 points and all other alternatives only 1 point.

3. We say that there is strategic voting when individuals have incentives to vote falsely for an alternative that is not they favorite ones, i.e. when they have incentives not to truly reveal their preferences. Show that there is some scope for strategic voting under plurality voting.

Let us consider every voter and try to see whether she has any incentive to vote for an alternative that is not her preferred one.

Voter 1 has no incentives to vote for another option than a as she is strictly better off under a than under b. Voters 2 and 4 have no incentives to vote for another option than b as this is their preferred option that will be chosen under plurality voting. Voters 3 and 5's preferred alternatives won't be selected. However, they would be better off under a than under b. So, they have incentives to vote for a rather than for d and c, respectively.

Note that we make the strong implicit assumption that voters know each other's preferences.

4. Forget about strategic voting. Let us now consider voting using a Borda rule. Which option will be selected if the Borda rule assigns 4 points to the preferred option, 3 points to the second one, 2 points to the third one, and 1 point to the last one?

Let us note $\mathbb{B}(i)$ the Borda score of alternative i. We get: $\mathbb{B}(a) = 16$, $\mathbb{B}(b) = 15$, $\mathbb{B}(c) = 11$, and $\mathbb{B}(d) = 8$. So, option a will be chosen under this Borda rule system.

5. Assume that you are voter 2 and that you are in charge of designing to Borda weights. How can you design the rule such as to be better off?

Voter 2 would like to see option b chosen. Note that we can write the Borda score of alternative i as:

$$\mathbb{B}(i) = \sum_{j=1}^{j=4} w_j \times \# \left\{ i = j^{\text{th}} \right\}$$

Let us note $w_1 = w$, $w_2 = x$, $w_3 = y$, and $w_4 = z$. We get:

$$\begin{aligned}
\mathbb{B}(a) &= w + 4x, \\
\mathbb{B}(b) &= 2w + x + 2y, \\
\mathbb{B}(c) &= 3y + w + z, \\
\mathbb{B}(d) &= w + 4z.
\end{aligned}$$

So, what we need to do is to find $\{w, x, y, z\}$ such that $w > x > y > z$ and $\mathbb{B}(b) > \max\{\mathbb{B}(a), \mathbb{B}(c), \mathbb{B}(d)\}$. These conditions can be rewritten as:

$$\begin{cases}
2w + x + 2y > w + 4x \\
2w + x + 2y > 3y + w + z \\
2w + x + 2y > w + 4z \\
w > x > y > z
\end{cases}$$

Let us set $z = 0$. The system can be rewritten as:

$$\begin{cases} w + 2y > 3x \\ w + x > y \\ w + x > -2y \\ w > x > y > 0 \end{cases}$$

As the third condition will always be satisfied as long as $y > 0$, we get:

$$\begin{cases} w + 2y > 3x \\ w + x > y \\ w > x > y > 0 \end{cases}$$

Let us set $y = 1$, we get:

$$\begin{cases} w + 2 > 3x \\ w + x > 1 \\ w > x > 1 \end{cases}$$

As the second condition will always been satisfied as long as $x > 1$, we get:

$$\begin{cases} w > 3x - 2 \\ x > 1 \end{cases}$$

By setting $x = 2$, we get $w > 4$. So, choosing $\{w, x, y, z\} = \{5, 2, 1, 0\}$ should do the job. Indeed, this weighting design provide us with $\mathbb{B}(a) = 13$, $\mathbb{B}(b) = 14$, $\mathbb{B}(c) = 11$, and $\mathbb{B}(d) = 5$.

Exercise 2

Sequential pairwise voting is a voting method that consists in a pre-determined sequence of pairwise confrontations. This pre-determined sequence is called a "fixed agenda". The chosen alternative is the one that survives the last pairwise majority vote.

1. Demonstrate that if a Condorcet winner exists, then this alternative will be the chosen one for any possible sequential pairwise voting procedure.

A sequential pairwise voting procedure is a pre-determined sequence that start with a face-to-face between two options and continue as long as needed with the introduction of others options against the winner of the preceding one. Therefore, choosing the pre-determined sequence amounts to choose at which stage each alternative will be introduced. If a Condorcet winner exists, then no matter at which stage it is introduced as it will always defeat all its potential opponents. So, the Condorcet winner will be the chosen alternative for any possible sequential pairwise voting procedure.

Let us consider the following profiles of preferences over 4 alternatives $\{a, b, c, d\}$ by 3 voters:

	Voter 1	Voter 2	Voter 3
Most preferred option	a	c	b
	b	a	d
	d	b	c
Least preferred option	c	d	a

In other terms, preferences of voter 1 are such that: $a \succ_1 b \succ_1 d \succ_1 c$.

2. Check that there is no Condorcet winner in the above-described situation.

Here is the list of all possible face-to-face situations and their implied social choice:

$$\begin{aligned} a \text{ versus } b &\implies a \succ b \\ a \text{ versus } c &\implies c \succ a \\ a \text{ versus } d &\implies a \succ d \\ b \text{ versus } c &\implies b \succ c \\ b \text{ versus } d &\implies b \succ d \\ c \text{ versus } d &\implies d \succ c \end{aligned}$$

Accordingly, none of the alternatives wins over all others in pairwise majority votes. So, there is no Condorcet winner.

3. Show that is sequential pairwise voting is used and if your have the agenda-setting power, then you can arrange the sequence to ensure whichever alternative you want to be chosen.

Assume you want alternative a to be selected. By looking at the series of outcomes presented in the previous question, we know that a would survive against b and d, but not against c. So, we need that the last face-to-face is either $a - -b$ or $a - -d$. Similarly, we need c to be eliminated before a is introduced. It turns out that c can be defeated by b. So, a sequence that will make a the selected alternative is as follows: start with c against b, introduce d, introduce a. Using a similar reasoning, a fixed agenda that will allow to select b is: start with a against c, introduce b, introduce d. To select c: start with b against d, then the winner against a, and then the winner against s. Finally, to select d, start with a against b, introduce c, introduce d.